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ON THE FORCE DECOMPOSITIONS OF LIGHTHILL AND MORISON

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Lighthill's assertion that the viscous drag force and the inviscid inertia force acting on a bluff body immersed in a time-dependent flow operate independently is not in conformity with the existing exact solutions and experimental facts. The two force components are interdependent as well as dependent on the parameters characterizing the phenomenon: the rate of diffusion of vorticity, relative amplitude of the oscillation, and the surface roughness.

1. INTRODUCTION

BATCHELOR (1967) STATED THAT "One of the most important problems of fluid mechanics is to determine the properties of the flow due to moving bodies of simple shape, over the entire range of values of Re, and more especially for the large values of Re corresponding to bodies of ordinary size moving through air and water." Advances in fluid mechanics over the past thirty years have only enhanced the importance of the problem. Separated flows in general and time-dependent flows in particular arise in many engineering situations, and the prediction of the fluid/structure interaction (forces and dynamic response) presents monumental mathematical, numerical and experimental challenges in both laminar and turbulent flows. Among them, sinusoidally oscillating flow about a circular cylinder or the sinusoidal motion of a cylinder in a viscous fluid otherwise at rest has been of special interest to offshore engineers for at least half a century.

Morison *et al.* (1950) proposed a force decomposition, for the determination of wave force on a "cylindrical object", which may be written as

$$F(t) = \frac{1}{2}\,\rho C_d D|U|U + \rho C_m \frac{\pi D^2}{4}\frac{\mathrm{d}U}{\mathrm{d}t}, \tag{1}$$

where $F(t)$ represents the in-line force acting on a cylinder as a linear sum of a velocity-squared-dependent drag force and an acceleration-dependent inertial force. The coefficients C_d and C_m are experimentally determined, cycle-averaged drag and inertia coefficients, over a broad range of the governing parameters: the Keulegan–Carpenter number K $(=U_m T/D)$, Reynolds number Re $(=U_m D/\nu)$ or β $(=\mathrm{Re}/K = D^2/\nu T)$, and the relative roughness k/D, where D is the diameter of the circular cylinder, U_m the maximum velocity in a cycle, T the period of flow oscillation, ν the kinematic viscosity of fluid, and k the mean roughness height. The force decomposition of Morison *et al.* (1950) is semi-empirical and its justification is strictly pragmatic and rests with experimental confirmation. It does not perform uniformly well in all ranges of K, β, and k/D [see e.g. Sarpkaya (1977, 1981, 1986, 1992)]. Comprehensive data on C_d and C_m have been presented by Sarpkaya (1976, 1986, 1999*a*).

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The generalized formulations of the force exerted on a rigid body in translational motion in an incompressible viscous fluid require either the velocity or the vorticity field to be known throughout the whole fluid space [see e.g. Wu (1981), Quartapelle & Napolitano (1983), Lighthill (1986*a*), and Howe (1989)]. Wu (1981) and Lighthill (1986*a*) expressed the force $\mathbf{F}$ in terms of the moment of the vorticity distribution over the entire space, as

$$\mathbf{F} = -\sigma\rho \frac{\partial}{\partial t} \int \mathbf{x} \wedge \boldsymbol{\omega} \, \mathrm{d}^3\mathbf{x} + \rho V_b \frac{\partial U}{\partial t}, \tag{2}$$

where $\sigma = \frac{1}{2}$ for three-dimensional flows and V_b is the volume of the body. According to equation (2), the first term represents the total contribution of vorticity and the second term the inviscid inertial force. This formulation has been adopted by Lighthill (1986*b*) for a similar decomposition of the in-line force acting on a cylinder in time-dependent flow (to be discussed later).

Howe (1989) has shown that the force F_i exerted by the fluid on a rigid body in the i-direction is given by

$$F_i = -A_{ij} \frac{\partial U_j}{\partial t} + \rho \int \nabla\chi_i \cdot (\boldsymbol{\omega} \wedge \mathbf{v}_{\mathrm{rel}}) \, \mathrm{d}^3\mathbf{x} + \mu \oint_S (\boldsymbol{\omega} \wedge \nabla\chi_i) \cdot \mathrm{d}\mathbf{S}, \tag{3}$$

in which A_{ij} is the added mass tensor of the body for translational motion, $\mathbf{v}_{\mathrm{rel}}$ $(=v - U)$ is the relative velocity, χ_i is the velocity potential of the irrotational flow about the body, $\mathrm{d}\mathbf{S}$ is the elemental surface area, and $\boldsymbol{\omega}$ is the vorticity. It is seen that F_i is represented as the sum of its three constituent components: an inviscid inertial force, *a vector sum of the normal surface stresses induced by vorticity in the fluid*, and a shear force or skin friction. Note that the second component is of particular importance for the purpose of this note.

2. STOKES CANONICAL SOLUTIONS

Stokes (1851) presented the solutions for both a sphere and cylinder oscillating in a liquid with the velocity $U = -A\omega \cos \omega t$, assuming that the amplitude A of the oscillations is small and the flow about the bodies is laminar, unseparated, and stable. For a sphere, his solution may be written as

$$F(t) = \left(\frac{1}{2} + \frac{9}{2}(\pi\beta)^{-1/2}\right) \frac{\rho\pi D^3}{6} \frac{\mathrm{d}U}{\mathrm{d}t} + \left(1 + \frac{1}{2}(\pi\beta)^{1/2}\right)(3\pi\mu D U), \tag{4}$$

which shows that both the drag and inertia force are modified by the Stokes number β, expressing the rate of diffusion. In other words, it is impossible to decompose $F(t)$, for the flow under consideration, into an inviscid inertia force and a viscous force. Both are affected by the diffusion of vorticity in which resides the memory of viscous fluids. If diffusion has sufficient time to adjust to the unsteady conditions imposed on the flow, it may be said that the motion is a juxtaposition of steady states or a slowly varying unsteady flow. If the diffusion cannot adjust to the conditions imposed on the flow, each succeeding state will be increasingly affected not only by the prevailing conditions but also by the past history of the motion. Equation (4) shows that, the inertia force approaches its ideal value only when the rate of diffusion β $[=(D^2/\nu)(1/U_m)(\mathrm{d}U/\mathrm{d}t)_0]$ increases to very large values while maintaining unseparated, stable, laminar flow, i.e. the conditions for the validity of equation (4).

Stokes' solution for a circular cylinder, as later extended to higher terms by Wang (1968), may be decomposed into in-phase and out-of-phase components as

$$C_a = 1 + 4(\pi\beta)^{-1/2} + (\pi\beta)^{-3/2}, \tag{5a}$$

$$C_d = \frac{3\pi^3}{2K}\left[(\pi\beta)^{-1/2} + (\pi\beta)^{-1} - \frac{1}{4}(\pi\beta)^{-3/2}\right], \tag{5b}$$

which shows the dependence of C_d and C_m on β and the fact that

$$\frac{C_a - 1}{KC_d} = \frac{8}{3\pi^3} \tag{6}$$

in the limit as $K \to 0$, $\beta \to \infty$.

Stokes' classical solutions formed the basis of many subsequent models where the oscillations are presumed to be small enough to allow convective accelerations to be ignored. Recently, Coimbra & Rangel (1998) presented the general solution of the particle momentum equation (Maxey & Riley 1983) for unsteady Stokes flows. Their solution "is only valid for small particle Reynolds number ($\mathrm{Re}_p = |U - V|a/\nu \ll 1$), small shear Reynolds number ($\mathrm{Re}_s = U_o a^2/\nu L \ll 1$) and applicable only for a small particle so that $a/L \ll 1$", where V and U are the particle and fluid velocities, respectively, a is the radius of the particle, and L is the characteristic length of the background flow. The case of finite particle Reynolds number requires appropriate corrections to the analytical solutions through higher-order expansions. An extensive review of the existing force models has shown that the degree of empiricism increases with increasing Reynolds number and some measure of the unsteadiness of the motion. The practice of expressing the time-dependent fluid force as a function of the prevailing velocities and accelerations has been extended to nonlinear motions where convective accelerations, separation, and three-dimensional wakes are important. Some of these efforts expressed the force as a sum of the quasi-steady component (a function of the body shape and the instantaneous Reynolds number), an ideal inertia component, and a history term (Basset 1888). A comprehensive discussion and numerical solution of the unsteady Navier–Stokes equations have been presented by Mei (1994) for flow over a stationary sphere at finite Reynolds numbers (Re < 100) with oscillating free-stream velocity in the range of $K < \mathrm{Re} < K^{-1}$. Mei's (1994) and Coimbra & Rangel's (1998) results are particularly illuminating and useful for particle dynamics where the Reynolds numbers are at least three orders of magnitude smaller than those encountered in the applications of the Morison equation.

3. LIGHTHILL'S FORCE DECOMPOSITION

The inertia and drag coefficients in the Morison model are forced to share the contributions of the vorticity field as in equations (4) and (5). Thus, the question naturally arises as to why one should not express the resistance in time-dependent flows as a sum of the contributions of (a) an inviscid inertial force (with a precisely determinable inertia coefficient, unlike that in Morison's equation) and (b) a vorticity–drag (in-line components of skin friction and the form drag) due to concentrated and distributed vorticity shed during the entire history of the motion (expressed, if at all possible, in terms of a single coefficient, dependent on K, Re, and k/D). In fact, Lighthill (1986*b*), following his equation (2), asserted that the viscous drag force and the inviscid inertia force operate independently and therefore it is possible to divide the measured time-dependent force into two distinct components: an inviscid inertial force, either corrected or uncorrected for a weakly nonlinear flow field (Madsen 1986), and a viscous drag force. Lighthill's assertion is based on Kelvin's minimum energy theorem which *assumes* that the motion of the unbounded external fluid may be expressed as a linear sum of (a) the potential flow that satisfies the boundary conditions, and (b) a residual vortex motion which satisfies the zero boundary conditions (at the interface and at infinity).

This led Lighthill (1986*b*) to suggest that " ... the irrotational part (a) of the fluid motion depends only upon those boundary conditions which it satisfies instantaneously". "This is the part [unlike (b), the vortex motion] which is devoid of any "memory" for earlier values taken by $U(t)$; rather, it is proportional simply to the current value of U, and its kinetic energy is proportional to U^2, ... so that $\frac{1}{2} M_a U^2$ can be thought of as if it were the kinetic energy of an added mass M_a of fluid which the body's motion effectively drags along with it." Then he goes on to state that "Simultaneously, the kinetic energy of part (b), the vortex motion, is increasing as more and more vorticity is shed into the wake, where the vortex lines are subsequently convected and diffused. The rate of working by the thrust with which the body acts upon the fluid is necessarily equal to the rate of increase of the total energy of the fluid; including (it must be emphasized) both the kinetic energy of part (b) and any thermal energy into which viscous dissipation may progressively convert that kinetic energy." However, Lighthill does not associate the increase in kinetic energy of part (b) with any added mass, as he has done so with that of part (a) as if the fluid "knew" how to differentiate the two increases in kinetic energy (one due to U and the other due to $\mathrm{d}U^2/\mathrm{d}t$. In fact he goes on to state that "An estimate of the rate of increase of energy in part (b) may be derived from the rate, proportional to ρAU (where A is the body's frontal area), at which the mass of wake fluid is growing. Velocities of the vortex motion are proportional to U, giving a rate of increase of energy $\frac{1}{2}\rho AU^3\, C_d$, where C_d is a coefficient. The corresponding thrust required to yield this rate of working, and to overcome the equal and opposite vortex-flow drag of the fluid on the body, is $\frac{1}{2}\rho AU^2\, C_d$. "This implies that the drag force is proportional to the square of the instantaneous velocity only, with no effect of the time rate of change of the kinetic energy as if the vorticity field were comprised of inviscid line vortices of constant strength, devoid of diffusion ($\beta \to \infty$) [a detailed discussion of this is given in Sarpkaya (1963, 1996)]".

Lighthill (1986*b*), using equation (2), re-wrote the Morison equation as

$$F = C_m^* \rho \dot{U} V_b + \tfrac{1}{2}\, \rho A_p U^2 C_d, \tag{7}$$

which, for an ambient flow defined by $U(t) = -\, U_m \cos \omega t$, reduces to

$$C_F = -\, C_d |\cos \omega t| \cos \omega t + C_m^* \frac{\pi^2}{K} \sin \omega t, \tag{8}$$

where C_m^* is now the ideal value of the inertia coefficient ($C_m^* = 2$ for a circular cylinder), and A_p and V_b are the projected area and the volume of the body, respectively. Lighthill's version of the Morison equation requires only one experimentally determined coefficient: C_d, presumably dependent on such parameters as the Reynolds number, Keulegan–Carpenter number, relative roughness, and direction of the body motion. Clearly, equations (7) and (8) make the entire K spectrum inertia-dominated, not just the region of relatively small K values. We will now show that it is impossible to find a suitable C_d value which enables equation (8) to represent the measured force even for a circular cylinder.

Figure 1 shows a representative measured force for a sinusoidally oscillating flow about a circular cylinder ($K \approx 15$, $\mathrm{Re} \approx 40\,000$). Also shown in this figure are the traces of force calculated using equation (8), with a C_d value that makes the maximum measured and calculated forces nearly agree, and the difference between the measured and calculated forces (the residue). This figure as well as several thousand others show conclusively that it is impossible to represent the measured force with equation (8), regardless of what value one

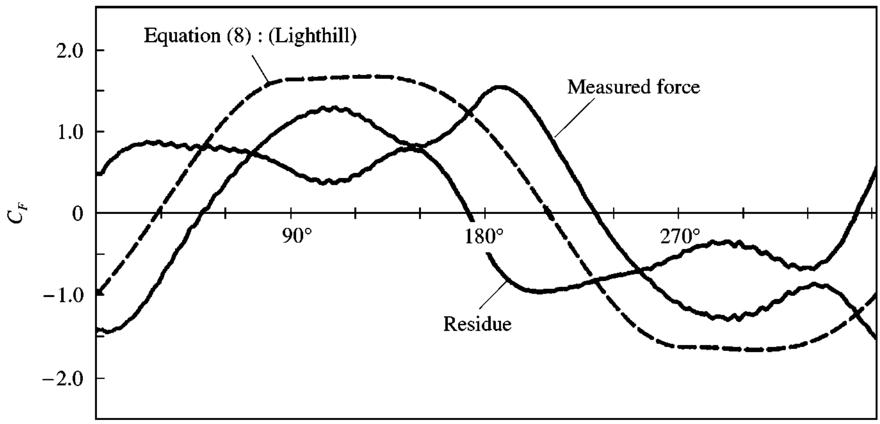

Figure. 1. Normalized measured force, force calculated from equation (8) due to Lighthill (1986*b*), and the residue = (calculated − measured) force.

assigns to C_d, as long as C_m^* (here equal to 2·0) is used. Only for very small values of K [where the second term in equation (8) is dominant] do the measured and calculated forces approach each other. The form of the residue immediately suggests, as verified by a proper Fourier analysis, that equation (8) must, for a first-order reduction of the residue, be augmented by a term involving sin θ (proportional to the acceleration of the flow). Using the C_d value deduced from a Fourier analysis of the measured force as the most appropriate drag coefficient, one has

$$C_F = -\, C_d|\cos \omega t|\cos \omega t + C_m^* \frac{\pi^2}{K} \sin \omega t - \frac{(C_m^* - C_m)\,\pi^2}{K} \sin \omega t, \tag{9}$$

which is obviously identical to the Morison equation. Here C_m is identical to the inertia coefficient C_m in the original Morison equation and is obtained in exactly the same manner as before. It must be emphasized that the use of a smaller or larger C_d could not have provided a better fit to the measured force. Furthermore, it would have required additional coefficients resulting from the expansion of C_d into a suitable series.

4. CLOSING REMARKS

It has been shown that the viscous drag force and the inviscid inertia force do not operate independently and it is not possible to divide the measured time-dependent force into an inviscid inertial force and a viscous drag force. Our results have shown convincingly that the creation, convection, and diffusion of vorticity affect both components of the force, because the unsteady flow is neither a juxtaposition of steady-flow states nor a juxtaposition of impulsively started unsteady-flow states. The inertia coefficient (or the added mass coefficient) varies with time during a given cycle and, if Fourier averaged, with the governing parameters (Re, K, k/D, shape and orientation of the body). It is now clear that the second term in equation (3), i.e. *the vector sum of the normal surface stresses induced by vorticity in the fluid*, contributes to both the inertial force and the drag or, in other words, modifies the ideal inertial force and the velocity-squared-dependent form drag and the skin-friction drag. Lighthill's decomposition [equations (7, 8)] is valid only for $K \to 0$, $\beta \to \infty$. It must be emphasized that what is questioned here is not the validity of the total force given by equation (2) but rather the decomposition of the total force into the constituents given by equation (8) and the assumptions of Kelvin and Lighthill leading to it.

Even though the foregoing has been presented in the context of in-line forces acting on bluff bodies, it is equally applicable to transverse forces acting on bluff bodies undergoing vortex-induced oscillations as evidenced by equally extensive data (Sarpkaya 1995). It appears from the foregoing that the Morison force decomposition does perform well over a broad range of K, β, and k/D values in spite of its well-known shortcomings. Recently, Sarpkaya (1999*b*) modified the Morison model to enhance its accuracy in the drag/inertia-dominated regime through the use of a third term, expressed in terms of the existing two coefficients and a new parameter based on $(\pi^2/K)(C_m^* - C_m)$.

Paul Adrien Maurice Dirac once noted that "A theory with mathematical beauty is more likely to be correct than an ugly one that fits some experimental data." Apparently, Morison's equation with Fourier-averaged empirical coefficients will remain as an exception to Dirac's wisdom as long as the concerns expressed by Batchelor (1967) remain theoretically unresolved.

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